

Extending the D'Alembert Solution to Space-Time Modified Riemann-Liouville Fractional Wave Equations

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April 3, 2012*

In the realm of complexity, it is argued that adequate modeling of TeV-physics demands an approach based on fractal operators and fractional calculus (FC). Non-local theories and memory effects are connected to complexity and the FC. The non-differentiable nature of the microscopic dynamics may be connected with time scales. Based on the Modified Riemann-Liouville definition of fractional derivatives, we have worked out explicit solutions to a fractional wave equation with suitable initial conditions to carefully understand the time evolution of classical fields with a fractional dynamics. First, by considering space-time partial fractional derivatives of the same order in time and space, a generalized fractional D'Alembertian is introduced and by means of a transformation of variables to light-cone coordinates, an explicit analytical solution is obtained. To address the situation of different orders in the time and space derivatives, we adopt different approaches, as it will become clear throughout this paper. Aspects connected to Lorentz symmetry are analyzed in both approaches.

PACS numbers: 05.45.-a, 11.10.Kk, 11.10.Lm

Keywords: coarse-grained, fractional calculus, fractional field theory, modified Riemann-Liouville

I. INTRODUCTION

During the last decade the interest of physicists in non-local field theories has been steadily increasing. The main reason for this development is the expectation, that the use of these field theories will lead to a much more elegant and effective way of treating problems in particle and high-energy physics, as it is possible up to now with local field theories. A particular subgroup of non-local field theories is described with operators of fractional nature and is specified within the framework of fractional calculus. Fractional calculus provides us with a set of axioms and methods to extend the concept of a derivative operator from integer order n to arbitrary order α , where α is a real value. Problems involving fractional integrodifferential is an attractive framework that in recent years awakened the interest of some researchers in the study on the fractional dynamics in many fields of physics such as in anomalous diffusion[1], mechanics, engineering and other areas [2, 3] and to deal with complex systems.

We can describe a complex system, as an 'open' system involving 'nonlinear interactions' among its subunits

which can exhibit, under certain conditions, a marked degree of coherent or ordered behavior extending well beyond the scale or range of the individual subunits. Usually it consists of a large number of simple members, elements or agents, which interact with one another and with the environment, and which have the potential to generate qualitatively "new" collective behavior, the manifestations of this behavior being the spontaneous creation of new spatial, temporal, or functional structures. Within the realm of complexity, new questions in fundamental physics have been raised, which cannot be formulated adequately using traditional methods. Consequently a new research area has emerged, which allows for new insights and intriguing results using new methods and approaches.

The interest in fractional wave equations aroused in 2000, with a publication by Raspini [4]. He demonstrated that a 3-fold factorization of the Klein-Gordon equation leads to a fractional Dirac equation which contains fractional derivative operators of order $\alpha = 2/3$ and, furthermore, the resulting γ - matrices obey an extended Clifford algebra [5], let us quote that, in the same year, the integer case was also studied in the work of Ref. [6].

It is well known that most of physical systems might be described by Lagrangian functions and that frequently their dynamical variables are on first order derivatives. This occurs probably because classical and quantum theories with superior order derivatives do not present a

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lower limit to the energy. Besides, the quantum field theories containing superior derivatives of fields usually present states with negative norm or ghost states and consequently a costly price is paid with the lost of unitarity violation [7].

The quantum dynamics of a system can be described by operators acting on a state vector, this is a good reason to study operators algebras and its unfolds. The language of operators is very useful and important to the physicists, we know today that the quantum analog of derivatives like, for instance, $\partial/\partial t$ and $\partial/\partial x$ are operators. We still know that derivatives with respect to q_k and p_k can be represented by equations like Poisson brackets and its quantum analog are commutators involving operators. The modern theory of pseudo-differential operators took its shape in the sixties, but we can consider the thirties, because the quantization problem, preliminarily solved by Weyl, and since the 1980's this tool has also yielded many significant results in nonlinear partial differential equations PDE.

Recently, Guy Jumarie [8] proposed a simple alternative definition to the Riemann-Liouville derivative. His modified Riemann-Liouville derivative has the advantages of both the standard Riemann-Liouville and Caputo fractional derivatives: it is defined for arbitrary continuous (non differentiable) functions and the fractional derivative of a constant is equal to zero. Jumarie's calculus seems to give a mathematical framework for dealing with dynamical systems defined in coarse-grained spaces and with coarse-grained time and, to this end, to use the fact that fractional calculus appears to be intimately related to fractal and self-similar functions.

We would like to stress that the choice of Jumarie's approach, besides the points already mentioned, is justified by the fact that chain and Leibnitz rules acquires a simpler form, which helps a great deal if changes of coordinates are performed. Moreover, causality seems to be more easily obeyed in a field-theoretical construction if we adopt Jumarie's approach.

As pointed out by Jumarie, non-differentiability and randomness are mutually related in their nature, in such a way that studies in fractals on the one hand and fractional Brownian motion on the other hand are often parallel in the same paper. A function which is continuous everywhere but is nowhere differentiable necessarily exhibits random-like or pseudo-random-features, in the sense that various samplings of this functions on the same given interval will be different. This may explain the huge amount of literature which extends the theory of stochastic differential equation to stochastic dynamics driven by fractional Brownian motion.

The most natural and direct way to question the classical framework of physics is to remark that in the space of our real world, the generic point is not infinitely small (or thin) but rather has a thickness. A coarse-grained space is a space in which the generic point is not infinitely thin, but rather has a thickness; and here this feature is modeled as a space in which the generic increment is not

dx , but rather $(dx)^\alpha$ and likewise for coarse grained with respect to the time variable t .

It is noteworthy, at this stage, to highlight the interesting work by Nottale [11], where the notion of fractal space-time is first introduced.

In this work we claim that the use of an approach based on a sequential form of Jumarie's Modified Riemann-Liouville [8] is adequate to describe the dynamics associated with fields theory and particles physics in the space of non-differentiable solution functions or in the coarse-grained space-time. Some possible realizations of fractional wave equations are given and the proposed solutions are analyzed. Based on this approach, we have worked out explicit solutions to a fractional wave equation with suitable initial conditions to carefully understand the time evolution of classical fields with a fractional dynamics. This has been pursued in (1+1) dimensions where the adoption of the light-cone coordinates allow a very suitable factorization of the solution in terms of left-and-right-movers. First, by considering space-time partial fractional derivatives of the same order in time and space, a generalized fractional D'Alembertian is introduced and by means of a transformation of variables to light-cone coordinates, an explicit analytical solution is obtained. Next, we address to the problem of different orders for time and space derivatives. In this situation, two different approaches have been adopted: one of them takes into account a non-differentiable space of solutions, whereas the other one considers a coarse-grained space-time as non-differentiable. For the former, there emerges an indicative of chiral symmetry violation. The latter points to a solution that depends on both the space and time orders of the derivatives. Aspects of Lorentz transform and invariance conditions are also analyzed. It is important to note that we are not assuming the validity of the semi-group property of the fractional derivatives [4] and are not working with generalized functions in the distributions sense [9] nor taking fractional power of operator [10]. Also, the solution technique here proposed does not make uses of Fourier transform and not necessarily Laplace transform. The work is organized as follows: After this Introduction, the Jumarie's modified fractional derivatives are briefly presented in Section 2. In the Section 3, the fractional wave equation is presented in the coarse-grained space-time and in non-differentiable function space of solutions. In Section 4 Lorentz transform and invariance conditions are analyzed. Following, in Section 5 we present an example. Finally, in Section 6, we cast our Concluding Comments. Two Appendices follow. The Appendices A and B treat respectively the aspects of calculation for different exponents and details of Lorentz transforms.

II. JUMARIE'S FRACTIONAL CALCULUS AND MODIFIED RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE

The well-tested definitions for fractional derivatives, so called Riemann-Liouville and Caputo have been frequently used for several applications in scientific periodic journals. In spite of its usefulness they have some dangerous pitfalls. For this reason, recently it was proposed another definition for fractional derivative [8], so called Modified Riemann-Liouville (MRL) fractional derivative and its basic definition is

$$\begin{aligned} D^\alpha f(x) &= \lim_{h \rightarrow 0} h^{-\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x + (\alpha - k)h) = \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} (f(t) - f(0)) dt; \\ 0 < \alpha < 1. \end{aligned} \quad (1)$$

Some advantages can be cited, first of all, using the MRL definition we found that derivative of constant is zero, and second, we can use it so much for differentiable as non differentiable functions. They are cast as follows:

(i) *Simple rules:*

$$\begin{aligned} D^\alpha K &= 0, \\ Dx^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \quad \gamma > 0, \\ (u(x)v(x))^{(\alpha)} &= u^{(\alpha)}(x)v(x) + u(x)v^{(\alpha)}(x). \end{aligned} \quad (2)$$

(ii) *Simple Chain Rules:*

$$\frac{d^\alpha}{dx^\alpha} f[u(x)] = \frac{d^\alpha f}{du^\alpha} \left(\frac{du}{dx} \right)^\alpha. \quad (3)$$

For non differentiable functions

$$\frac{d^\alpha}{dx^\alpha} f[u(x)] = \frac{df}{du} \frac{d^\alpha u}{dx^\alpha}. \quad (4)$$

for coarse-grained space-time.

For more details, the readers can follow the Ref. [12] which contains all the basic for the formulation of a fractional differential geometry in coarse-grained space, and refers to an extensive use of coarse-grained phenomenon.

Now that we have set up these fundamental expressions, we are ready to carry out the calculations of main interest: the solutions to our fractional wave equations

III. FRACTIONAL WAVE EQUATION IN THE MRL SENSE

Since the semi group properties for fractional derivatives in general does not hold, we used the Miller-Ross sequential derivative [13] in the MRL sense. Incidentally,

the Miller-Ross sequential derivative is a systematic procedure that carries out a fractional higher-order derivative while avoiding the recursive application of many single derivatives taken after each other. Moreover, we took the option to carry out the sequence of derivatives in the cascade form, in MRL sense, as done in the work of Ref. [14].

A. Coarse-grained space-time

Let us Suppose a class of transformations that maps from the variables x', t' to $u = \gamma x'^\alpha + \lambda \gamma t'^\beta$ to illustrate our prescription to apply the chain rule. Notice that the dimension of λ - *parameter* is such that $\lambda t'^\beta$ matches its dimension with x'^α .

Performing successive derivatives and using the chain rule (4), we can write

$$\frac{\partial^\alpha \phi}{\partial x'^\alpha} = \frac{d\phi}{du} \frac{\partial^\alpha u}{\partial x'^\alpha} = \frac{d\phi}{du} \gamma \Gamma(\alpha+1), \quad (5)$$

$$\frac{\partial^\alpha \partial^\alpha \phi}{\partial x'^\alpha \partial x'^\alpha} = \frac{d^2 \phi}{du^2} \gamma^2 \Gamma^2(\alpha+1), \quad (6)$$

$$\frac{\partial^\beta \phi}{\partial t'^\beta} = \frac{d\phi}{du} \frac{\partial^\beta u}{\partial t'^\beta} = \frac{d\phi}{du} \lambda \gamma \Gamma(\beta+1), \quad (7)$$

$$\frac{\partial^\beta \partial^\beta \phi}{\partial t'^\beta \partial t'^\beta} = \frac{d^2 \phi}{du^2} \lambda^2 \gamma^2 \Gamma^2(\beta+1). \quad (8)$$

Substituting Eq.(6) into the Eq. (8) yields:

$$\frac{\partial^\alpha}{\partial x'^\alpha} \frac{\partial^\alpha}{\partial x'^\alpha} \phi(x, t) - \frac{\Gamma^2(\alpha+1)}{\lambda^2 \Gamma^2(\beta+1)} \frac{\partial^\beta}{\partial t'^\beta} \frac{\partial^\beta}{\partial t'^\beta} \phi(x, t) = 0. \quad (9)$$

B. Non-differentiable functions

Supposing a class of transforms that maps from x', t' to $u = \gamma x' + v \gamma t'$ and using the chain rule (3), we can write

$$\frac{\partial^\alpha \phi}{\partial x'^\alpha} = \frac{d\phi}{du} \frac{\partial^\alpha u}{\partial x'^\alpha} = \gamma \frac{d^\alpha \phi}{du^\alpha} \left(\frac{\partial u}{\partial x'} \right)^\alpha = \gamma \frac{d^\alpha \phi}{du^\alpha}, \quad (10)$$

and

$$\frac{\partial^\alpha \partial^\alpha \phi}{\partial x'^\alpha \partial x'^\alpha} = \gamma^2 \frac{d^\alpha}{du^\alpha} \frac{d^\alpha \phi}{du^\alpha}. \quad (11)$$

Similarly, for a fractional time derivative, we have

$$\frac{\partial^\beta \phi}{\partial t'^\beta} = \frac{d^\beta \phi}{du^\beta} \left(\frac{\partial u}{\partial t'} \right)^\beta = \gamma \frac{d^\beta \phi}{du^\beta} (\pm v)^\beta, \quad (12)$$

and

$$\frac{\partial^\beta}{\partial t'^\beta} \frac{\partial^\beta \phi}{\partial t'^\beta} = \gamma^2 \frac{d^\beta}{du^\beta} \frac{d^\beta \phi}{du^\beta} (\pm v)^{2\beta}. \quad (13)$$

The connection can be done for $\alpha = \beta$, which leads to

$$\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \phi(x, t) - \frac{1}{v^{2\alpha}} \frac{\partial^\alpha}{\partial t^\alpha} \frac{\partial^\alpha}{\partial t^\alpha} \phi(x, t) = 0. \quad (14)$$

Now with t and x with the same power, v has the dimension of speed.

1. Fractional D'Alembertian of same space-time partial derivative order

The D'Alembertian can now be established as:

$$\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \phi(x, t) - \frac{1}{v^{2\alpha}} \frac{\partial^\alpha}{\partial t^\alpha} \frac{\partial^\alpha}{\partial t^\alpha} \phi(x, t) = 0, \quad 0 < \alpha < 1 \quad (15)$$

Proceeding now a variable change in a light cone, right and left movers, respectively, we can write

$$\begin{cases} \xi = x - vt \\ \eta = x + vt \end{cases}, \quad (16)$$

Assuming a non-differentiable space of solutions, the chain rule in the MRL sense(3) and using Eqs. (15), (16), after some algebraic manipulations, we obtain

$$\frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial^\alpha}{\partial \eta^\alpha} \phi(\xi, \eta) = 0. \quad (17)$$

In connection with wave equations modified to more complex structures, we would like to call attention to the work of Ref. [15] where the authors propose non linear wave equations with modified functions.

Solutions in the light-cone coordinates:

The form of the Eq. (15) suggests a solution of the form, as in the case of integer derivatives:

$$\tilde{\phi}(\xi, \eta) = f(\xi) + g(\eta), \quad (18)$$

subject to the initial conditions

$$\left\{ \begin{array}{l} \phi(x, 0) = F(x) \\ \frac{\partial^\alpha \phi(x, t)}{\partial t^\alpha} \Big|_{t=0} \equiv G_*(x) \end{array} \right. \quad (19)$$

According to the initial conditions,

$$F(x) = f(x) + g(x), \quad (20)$$

and applying the fractional derivative of order α to Eq. (20) yields:

$$f^{(\alpha)}(x) = F^{(\alpha)}(x) - g^{(\alpha)}(x). \quad (21)$$

By means of the second of Eqs.(18), (19) and using the chain rule Eq.(3), we obtain

$$g^{(\alpha)}(x) = \frac{G_*(x)}{v^\alpha} - (-\text{sgn}(v))^\alpha f^{(\alpha)}(x). \quad (22)$$

Now assuming v as a positive quantity and substituting Eq. (21) into Eq. (22), we can eliminate the $f^{(\alpha)}(x)$ dependence, resulting in

$$g^{(\alpha)}(x) = \frac{G_*(x)}{2} - (-1)^\alpha \frac{F^{(\alpha)}(x)}{2}. \quad (23)$$

To obtain $g(x)$ we fractional integrate the Eq. (23), considering as an initial value problem with initial conditions given by (19). This can be done, for example, by applying Laplace transform and its inverse. The result is

$$\begin{aligned} g(x) &= \frac{1}{2\Gamma(\alpha)} \int_0^x (x-\tau)^{(\alpha-1)} G_*(\tau) d\tau + \\ &+ \frac{(-1)^\alpha}{2} [F(0) - F(x)] + g(0) \end{aligned} \quad (24)$$

where $\Gamma(\alpha)$ is the gamma function.

We have then found the functional forms of f and g , so that the general solution for a general instant of time, t , can be expressed as below:

$$\begin{aligned} \phi(\xi, \eta) &= \frac{1}{2\Gamma(\alpha)} \int_\xi^0 (\xi-\tau)^{(\alpha-1)} G_*(\tau) d\tau + \\ &+ \frac{1}{2\Gamma(\alpha)} \int_0^\eta (\eta-\tau)^{(\alpha-1)} G_*(\tau) d\tau + \\ &+ F(\xi) \left[\frac{(-1)^\alpha}{2} + 1 \right] - F(\eta) \frac{(-1)^\alpha}{2}. \end{aligned} \quad (25)$$

2. Analysis of the exponents analysis for preservation of chirality:

To preserve the chiral symmetry we can choose the fractional exponents to satisfy $(-1)^\alpha = -1$. This condition can be written as

$$\exp i(\pi + 2n\pi)\alpha = \exp i(\pi + 2k\pi), \quad n, k \in \{\mathbb{N} \cup \{0\}\}, \quad (26)$$

which leads to condition for the fractional exponents that preserves chirality as

$$\begin{aligned} \alpha &= \frac{2k+1}{2n+1}, & 0 < \alpha < 1, \\ k &= 0, 1, 2, \dots; & n &= k+1, k+2, \dots \end{aligned} \quad (27)$$

3. Regularization to preserve chiral symmetry

In order to impose the preservation of chiral symmetry properties, we can define an regularized fractional derivative in the MRL sense as

Definition: $\partial^\alpha / \partial \hat{t}^\alpha \equiv (\text{sign}(MV_k))^\alpha \partial^\alpha / \partial t^\alpha$, where $\text{sign}(MV_k)$ is the signal of the mover left or right given by

$$MV_k = \begin{cases} -1, & k = \xi \text{ for left movers} \\ +1, & k = \eta \text{ for right movers} \end{cases} \quad (28)$$

With the above definition, the fractional wave equation could be written as

$$\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \phi(x, t) - \frac{1}{v^{2\alpha}} \frac{\partial^\alpha}{\partial \hat{t}^\alpha} \frac{\partial^\alpha}{\partial \hat{t}^\alpha} \phi(x, t) = 0 \quad (29)$$

and the solutions will not carry the complex factor, preserving so the chirality.

We also proceed similarly with an wave equation with different exponents in space and time

$$\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \phi(x, t) - \frac{1}{v^{2\beta}} \frac{\partial^\beta}{\partial t^\beta} \frac{\partial^\beta}{\partial t^\beta} \phi(x, t) = 0. \quad (30)$$

The development is in Appendix A.

The general result can also indicate that a regularized definition of derivative, similar to the Feller definition by using MRL derivative, could be more adequate to handle this kind of problem. Also a regularized definition could give a option to conserve the parity or the chiral properties of the field.

In the sequence we propose an alternative approach by considering fractional space-time instead of fractional space functions, that is, we consider that a coarse-grained space-time, meaning that nor the space nor the time are infinitely thine but have "thickness".

C. Coarse-grained space-time

We now consider the problem with a coarse-grained space-time which means that space and time are non-differentiable and considering the chain rule as [8]

$$\frac{d^\alpha}{dx^\alpha} f[u(x)] = \frac{d}{du} f \frac{d^\alpha}{dx^\alpha} u. \quad (31)$$

It can be shown that the ansatz $\phi = \phi(x^\alpha + \lambda t^\beta)$ is a solution of the fractional wave equation (30) in a coarse-grained space-time [8], subject to the condition

$$\lambda_{\alpha, \beta} = \pm v^\beta \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)}. \quad (32)$$

The result above gives the insight to redefine the light-cone variables ξ, η as

$$\begin{cases} \xi = x^\alpha - \lambda t^\beta \\ \eta = x^\alpha + \lambda t^\beta \end{cases}. \quad (33)$$

Applying the rule (31) to the fractional D'Alembertian (30) with the new variables (33), we obtain, after some algebra, a simple form

$$\frac{\partial^2}{\partial \xi \partial \eta} \phi(\xi, \eta) = 0, \quad (34)$$

subject to the condition (32). Eq. (34) permits to apply the same procedure of subsection (III B 1). The result is

$$\phi(\xi, \eta) = \frac{1}{2K\Gamma(\beta+1)} \int_\xi^\eta G_{**}(y^\alpha) dy + \frac{F(\xi)}{2} + \frac{F(\eta)}{2}, \quad (35)$$

where

$$\left. \frac{\partial^\beta \phi(x, t)}{\partial t^\beta} \right|_{t=0} \equiv G_{**}(x^\alpha) \quad (36)$$

The advantage of this approach is that there is no violation of chirality and open the perspective to study higher orders derivatives in fractional space-time by applying successively the operator given by Eq. (34) to an ansatz $\phi(\xi, \eta)$.

The introduction of higher derivatives yields the so-called negative squared-norm states ghosts. Here, we argue that the presence of fractional higher derivatives might remove the problem of these unphysical modes.

IV. LORENTZ TRANSFORMS AND INVARIANCE CONDITIONS

The standard Lorentz boosts in (1+1) dimensions read:

$$\begin{cases} x' = \gamma(x - vt) \\ t' = \gamma(-\frac{v}{c^2}x + t) \end{cases}, \quad (37)$$

Its inverse can be obtained by changing v by $(-v)$.

We shall obtain the fractional Lorentz transforms in a way which differs from Jumarie's approach. Considering now the fractional front wave as

$$\begin{aligned} c^{2\alpha}(t^\alpha)^2 - (x^\alpha)^2 - (y^\alpha)^2 - (z^\alpha)^2 &= \\ c^{2\alpha}(t'^\alpha)^2 - (x'^\alpha)^2 - (y'^\alpha)^2 - (z'^\alpha)^2 &, \end{aligned} \quad (38)$$

we suppose a fractional transformation of form

$$\begin{cases} x'^\alpha = \gamma_{\alpha,\beta}(x^\alpha - \lambda t'^\beta) \\ t'^\beta = \gamma_{\alpha,\beta}(t^\beta - \frac{\lambda}{c^{2\beta}} x^\alpha) \end{cases} \quad (39)$$

The inverse transform can directly be obtained as

$$\begin{cases} x^\alpha = \gamma_{\alpha,\beta}(x'^\alpha + \lambda t'^\beta) \\ t^\beta = \gamma_{\alpha,\beta}(t'^\beta + \frac{\lambda}{c^{2\beta}} x'^\alpha) \end{cases} \quad (40)$$

From Eq.(38), with the transformations above, we are lead to a fractional gamma factor which reads as below:

$$\gamma_{\alpha,\beta} = \frac{1}{\sqrt{1 - \frac{\lambda^2}{c^{2\beta}}}}. \quad (41)$$

From Eq. (32), we see that $\lambda_{\alpha,\beta}$ depends on the fractional exponents α, β .

A. Fractional Lorentz transform invariance for coarse-grained space-time

We have shown that a function of $\phi(x^\alpha, t^\beta)$ is a solution of wave equation given by Eq.(30). It can be shown that this wave equation is Invariant to Fractional Lorentz transform. The details are in Appendix B.

B. Standard Lorentz invariance for non-differentiable space of solutions

It can also be proved that in the space of non-differentiable solutions, the fractional wave equation is Lorentz invariant by standard Lorentz transforms, if exponents of fractional derivatives in space and time are equal to each other. The details can also be found in appendix B.

V. AN EXPLICIT EXAMPLE OF SOLUTION

As an illustrative example, let us take our initial conditions as $F(x) = \phi(x, 0) = \Theta(x)\Theta(1-x)$, where $\Theta(x)$ is the Heaviside function and, $\frac{\partial^\alpha \phi(x, t)}{\partial t^\alpha} \Big|_{t=0} \equiv G_*(x) = 0$.

With these conditions, the time evolution of the general solution for non-differential functions, acquires the form

$F(x) = \phi(x, 0) = \Theta(x)\Theta(1-x)$, and $\frac{\partial^\alpha \phi(x, t)}{\partial t^\alpha} \Big|_{t=0} \equiv G_*(x) = 0$. With these conditions, the time evolution of the general solution for non-differential functions, ac-

quires the form

$$\begin{aligned} \phi(x, t) &= F(x-vt) \left[\frac{(-1)^\alpha}{2} + 1 \right] - F(x+vt) \left[\frac{(-1)^\alpha}{2} + 1 \right] \\ \phi(x, t) &= \Theta(x-vt)\Theta(1-x+vt) \left[\frac{(-1)^\alpha}{2} + 1 \right] + \\ &\quad - \Theta(x+vt)\Theta(1-x-vt) \frac{(-1)^\alpha}{2}. \end{aligned} \quad (42)$$

The solution above represents two well localized propagating rectangular pulses, propagating in opposite directions, with different attenuation parameters that depend on the chirality and the fractional exponent. Again, if the fractional exponent is one of those that preserves the chiral symmetry, the solution is identical to the case of an integer exponent.

For the case of fractional space-time, the solutions is similar for this example but with different space and time scales with the chiral symmetry preserved.

VI. CONCLUDING REMARKS

In this work, by taking into account a space of non-differentiable functions in one case and a non-differentiable space-time (coarse-grained) in the other, we have obtained in a natural way fractional wave equations in terms of a fractional D'Alembertian with a sequential form of modified fractional Riemann-Liouville. We claim that the novelty of our work is the particular choice of light-cone coordinates along with the use of sequential modified fractional derivatives and an adequate chain rules, leading to technique that creates a perspective to obtain solutions for other similar problems. Our solutions are worked out for general initial conditions. The Jumarie's approach of fractional calculus seems to be more adequate to deal with problems that involve transformations in coordinates, since the chain and Leibniz rules are less complicated. Since we are choosing to work with non-differentiable functions or a coarse-grained space-time, no use of distributional generalized functions or fractional powers of operators, neither the maintenance of semi-group properties of exponents in the derivatives is made. In each case of study the results agrees with standard integer order in the convenient limits.

In terms of the non-differentiable space of solutions, we have constructed a wave equation and shown that the space and time orders of fractional derivatives must be the same in order to make physics sense. A solution obtained with the light-cone coordinates shows a possible violation of the chiral separation. The exponents are discriminated in order to distinguish cases to preserve the chiral properties. A suggestion to intentionally prevent the chiral violation is presented in terms of a movers sign regularized fractional derivatives.

In the non-differentiable space-time (coarse grained), the form of solutions to the wave equation in terms

of power of space and time, gives a path to understand the a fractional Lorentz transform proposed by Jumarie, in terms of metric invariant radius of a propagating fractional front wave. The approach used opened up possibilities to further studies of higher-fractional order wave equations. Complementary, we have explicitly shown that the fractional wave equation in terms of the non-differentiable space-time is invariant under a Lorentz transform-like called fractional Lorentz transform, within the conditions of equality for fractional orders of derivatives in space and in time. Similar results from fractional wave equation in a non-differentiable space of solutions functions in terms of standard Lorentz transform.

We should point out that, if Lorentz symmetry is not at work, the study of systems with a mismatch between the number of time and space derivatives is common in a number of condensed-matter systems, as described in the theory of dynamical critical systems and quantum criticality [17, 18]. The remarkable aspect of these systems is the presence of scaling properties that are anisotropic in time and space. Nowadays, the construction of gravitational models and field theories is intimately connected to the number of space-time dimensions, extra dimensions, as well as eventual effects of fractional dimensions and possible contributions from non-holonomic commutative/non-commutative variables of such fractal dimensions. We understand that it is important to seek alternative physical concepts by means of different approaches of calculus and geometry, considering new ideas and models for space-time [19].

More recently, based on the theory of Lifschitz scalars

in arbitrary dimensions [20], Horava introduced a new class of quantum gravity models [21] with the outstanding property of renormalisability in (1+3) dimensions. Use of FC in this context and related ones is indicated in the Calcagni's work [22] and references therein.

VII. ACKNOWLEDGMENTS

The authors J. Weberszpil and J. A. Helayël-Neto wish to express their gratitude to FAPERJ-Rio de Janeiro and CNPq-Brazil for the partial financial support.

Appendix A: Fractional D'Alembertian of different space-time partial derivative orders

Eq. (30) in light-cone coordinates take the form:

$$\begin{aligned} & \left[\frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial^\alpha}{\partial \xi^\alpha} - \frac{\partial^\beta}{\partial \eta^\beta} \frac{\partial^\beta}{\partial \eta^\beta} \right] \phi(\xi, \eta) + \\ & + 2 \left[\frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial^\alpha}{\partial \eta^\alpha} - (-1)^\beta \frac{\partial^\beta}{\partial \xi^\beta} \frac{\partial^\beta}{\partial \eta^\beta} \right] \phi(\xi, \eta) + \\ & + \left[\frac{\partial^\alpha}{\partial \eta^\alpha} \frac{\partial^\alpha}{\partial \eta^\alpha} - \frac{\partial^\beta}{\partial \xi^\beta} \frac{\partial^\beta}{\partial \xi^\beta} \right] \phi(\xi, \eta) = 0. \quad (\text{A1}) \end{aligned}$$

The result above is not so compact as the previous one. If $\alpha = \beta \neq 1$ we recover the previous result. If $\alpha = \beta = 1$, the result is consistent with the literature, see for example [16].

Now, back to the $(t; x)$ -coordinates, we get to an expression which is not clearly Eq. (30):

$$\left(\frac{1}{2} \right)^{\alpha+\beta} \left[\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial x^\beta} + \frac{1}{v^\beta} \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial t^\beta} + (-1)^\alpha \frac{1}{v^\alpha} \frac{\partial^\alpha}{\partial t^\alpha} \frac{\partial^\beta}{\partial x^\beta} + (-1)^\alpha \frac{1}{v^{\alpha+\beta}} \frac{\partial^\alpha}{\partial t^\alpha} \frac{\partial^\beta}{\partial t^\beta} \right] \phi(x, t) = 0. \quad (\text{A2})$$

Again, if $\alpha = \beta = 1$ we recover the traditional D'Alembertian; but, if we transform the above expression back to light-cone of variables, the result, due to the non-biunivocity of the functions is not the starting point, $\frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial^\beta}{\partial \eta^\beta} \phi(\xi, \eta) = 0$, but coincide with the literature if $\alpha = \beta = 1$. This can be understood if we imagine fractal space functions and try to go from one point into to others passing trough a point of ramification, characterizing the multiplicity of solutions.

Notice that in (A2), if we take $\alpha = \beta$, and if $(-1)^\alpha = -1$, chirality is preserved and we recover Eq. (30).

Appendix B: Fractional Lorentz transform invariance for coarse-grained space-time

Using the chain rule (31) and the inverse fractional Lorentz transforms given by (40), we obtain

$$\begin{aligned} \frac{\partial^\alpha \phi}{\partial x'^\alpha} &= \frac{\partial \phi}{\partial(x^\alpha)} \frac{\partial^\alpha x^\alpha}{\partial x'^\alpha} + \frac{\partial \phi}{\partial(t^\beta)} \frac{\partial^\alpha t^\beta}{\partial x'^\alpha} \\ \frac{\partial^\alpha \phi}{\partial x'^\alpha} &= \frac{\partial \phi}{\partial(x^\alpha)} \gamma \Gamma(\alpha + 1) + \frac{\partial \phi}{\partial(t^\beta)} \frac{\lambda \gamma \Gamma(\alpha + 1)}{c^{2\beta}}. \quad (\text{B1}) \end{aligned}$$

and using that

$$\begin{cases} \partial^\alpha \phi \cong \Gamma(\alpha + 1) \partial \phi \\ \partial^\beta \phi \cong \Gamma(\beta + 1) \partial \phi \end{cases}, \quad (\text{B2})$$

we can re-write Eq. (B1) as

$$\frac{\partial^\alpha \phi}{\partial x'^\alpha} = \gamma \frac{\partial^\alpha \phi}{\partial x^\alpha} + \frac{\lambda \gamma \Gamma(\alpha + 1)}{c^{2\beta} \Gamma(\beta + 1)} \frac{\partial^\beta \phi}{\partial t^\beta}. \quad (\text{B3})$$

Analogously, we can write for the fractional temporal derivative

$$\frac{\partial^\beta \phi}{\partial t'^\beta} = \gamma \frac{\partial^\beta \phi}{\partial t^\beta} + \frac{\lambda \gamma \Gamma(\beta + 1)}{\Gamma(\alpha + 1)} \frac{\partial^\alpha \phi}{\partial x^\alpha}. \quad (\text{B4})$$

Repeating this procedure for successive fractional derivatives, and assuming as a premise that the fractional wave equation is valid in the S referential, we show that it is valid in the S' referential as follows after some rearrangements

$$\begin{aligned} \frac{\partial^\alpha}{\partial x'^\alpha} \frac{\partial^\alpha \phi}{\partial x'^\alpha} - \frac{1}{v^{2\beta}} \frac{\partial^\beta}{\partial t'^\beta} \frac{\partial^\beta \phi}{\partial t'^\beta} &= \gamma^2 \left[\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha \phi}{\partial x^\alpha} - \frac{1}{v^{2\beta}} \frac{\partial^\beta}{\partial t^\beta} \frac{\partial^\beta \phi}{\partial t^\beta} \right] + \frac{2\lambda \gamma^2 \Gamma(\alpha + 1)}{c^{2\beta} \Gamma(\beta + 1)} \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta \phi}{\partial t^\beta} + \\ &- \frac{2\lambda \gamma^2 \Gamma(\alpha + 1)}{v^{2\beta} \Gamma(\beta + 1)} \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta \phi}{\partial t^\beta} - \left[-\frac{\lambda^2 \gamma^2 \Gamma^2(\alpha + 1)}{c^{4\beta} \Gamma^2(\beta + 1)} \frac{\partial^\beta}{\partial t^\beta} \frac{\partial^\beta \phi}{\partial t^\beta} + \frac{\lambda^2 \gamma^2 \Gamma^2(\beta + 1)}{v^{2\beta} \Gamma^2(\alpha + 1)} \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha \phi}{\partial x^\alpha} \right]. \end{aligned} \quad (\text{B5})$$

Using that $\alpha = \beta$ and $v^\beta = c^\beta$, we obtain that the LHS of (B5) is zero, that is, invariant for fractional Lorentz transforms.

$$\begin{aligned} \frac{\partial^\alpha \phi}{\partial x'^\alpha} &= \frac{\partial^\alpha \phi}{\partial x^\alpha} \left(\frac{\partial x}{\partial x'} \right)^\alpha + \frac{\partial^\alpha \phi}{\partial t^\alpha} \left(\frac{\partial t}{\partial x'} \right)^\alpha = \\ &= \frac{\partial^\alpha \phi}{\partial x^\alpha} \gamma^\alpha + \frac{\partial^\alpha \phi}{\partial t^\alpha} \left(\gamma \frac{v}{c} \right)^\alpha, \end{aligned} \quad (\text{B6})$$

Standard Lorentz invariance for non-differentiable space of solutions

Using the inverse of standard Lorentz transform (37) and the chain rule for non-differentiable functions (3) we can write

$$\begin{aligned} \frac{\partial^\beta \phi}{\partial t'^\beta} &= \frac{\partial^\beta \phi}{\partial x^\beta} \left(\frac{\partial x}{\partial t'} \right)^\beta + \frac{\partial^\beta \phi}{\partial t^\beta} \left(\frac{\partial t}{\partial t'} \right)^\beta = \\ &= \frac{\partial^\beta \phi}{\partial x^\alpha} (\gamma v)^\alpha + \frac{\partial^\beta \phi}{\partial t^\alpha} \gamma^\alpha. \end{aligned} \quad (\text{B7})$$

Proceeding again with the derivatives and assuming as a premise the validity of Eq.(30), we obtain that the fractional wave equations, in a space of non-differentiable solutions is Standard Lorentz invariant, as can be seen below

$$\begin{aligned} \frac{\partial^\alpha}{\partial x'^\alpha} \frac{\partial^\alpha \phi}{\partial x'^\alpha} - \frac{1}{c^{2\beta}} \frac{\partial^\beta}{\partial t'^\beta} \frac{\partial^\beta \phi}{\partial t'^\beta} &= \gamma^{2\alpha} \left[\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha \phi}{\partial x^\alpha} - \frac{1}{c^{2\beta}} \frac{\partial^\beta}{\partial t^\beta} \frac{\partial^\beta \phi}{\partial t^\beta} \right] + \frac{2\gamma^{2\alpha} v^\alpha}{c^{2\alpha}} \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha \phi}{\partial t^\alpha} - \frac{2\gamma^{2\beta} v^\beta}{c^{2\beta}} \frac{\partial^\beta}{\partial x^\beta} \frac{\partial^\beta \phi}{\partial t^\beta} + \\ &- \left[-v^{2\alpha} \gamma^{2\alpha} \frac{\partial^\alpha}{\partial t^\alpha} \frac{\partial^\alpha \phi}{\partial t^\alpha} + \gamma^{2\beta} \frac{v^{2\beta}}{c^{2\beta}} \frac{\partial^\beta}{\partial x^\beta} \frac{\partial^\beta \phi}{\partial x^\beta} \right], \end{aligned} \quad (\text{B8})$$

that is identically zero if $\alpha = \beta$ and $v^\beta = c^\beta$.

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